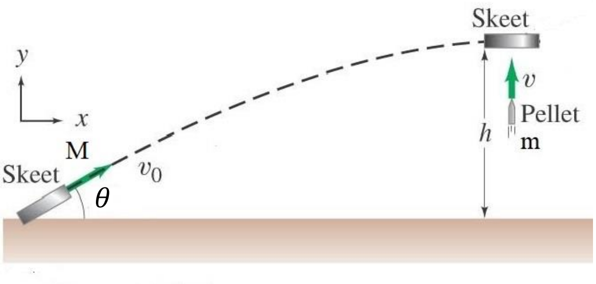
PHYS 101 Final Exam Solution 2020-21-2

PHYS 101 Final Exam Question 1

A skeet (clay target) with mass M is fired at an angle θ to the horizon with speed v_0. When it reaches the maximum height, h, it is hit from below by a pellet with mass m traveling vertically upward at speed v. The pellet is embedded in (sticks to) the skeet.

(a) (12 Pts.) What is the velocity of the skeet with the pellet embedded in it just after they collide?

(b) (8 Pts.) What is the velocity of the center of mass of the skeet-pellet system just before they collide?



Solution:

(a) Velocity of a projectile at maximum height is horizontal. Therefore, velocity of the skeet just before the collision with the pellet is

$$\vec{\mathbf{v}}_S = v_0 \cos\theta \ \hat{\mathbf{i}}\,.$$

Momentum of the skeet-pellet system just before the collision is

$$\vec{\boldsymbol{p}}_1 = \vec{\boldsymbol{p}}_S + \vec{\boldsymbol{p}}_P = M\vec{\mathbf{v}}_S + m\vec{\mathbf{v}}_P = Mv_0 \cos\theta \ \hat{\mathbf{i}} + mv\,\hat{\mathbf{j}}\,.$$

If we let $\vec{\mathbf{v}}'$ denote the velocity of the skeet with the pellet embedded in, momentum of the skeet-pellet system just after the collision is

$$\vec{\boldsymbol{p}}_2 = (M+m)\vec{\mathbf{v}}'.$$

Momentum is conserved in the collision. Therefore

$$\vec{\boldsymbol{p}}_2 = \vec{\boldsymbol{p}}_1 \quad \rightarrow \quad (M+m)\vec{\mathbf{v}}' = Mv_0\cos\theta \ \hat{\mathbf{i}} + mv\,\hat{\mathbf{j}} \quad \rightarrow \quad \vec{\mathbf{v}}' = \frac{Mv_0\cos\theta}{M+m}\,\hat{\mathbf{i}} + \frac{mv}{M+m}\,\hat{\mathbf{j}}\,.$$

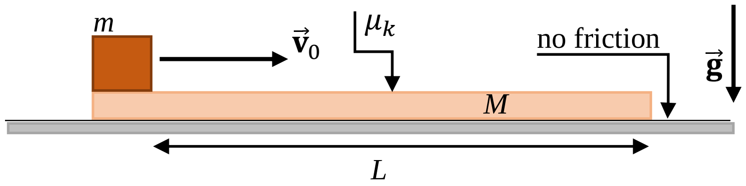
(b) Collision does not change the velocity of the center of mass. So it is equal to $\vec{\mathbf{v}}'$ found in part (a).

PHYS 101 Final Exam Question 2

A small block of mass m is placed on a board of mass M which itself rests on a frictionless horizontal surface as shown in the figure. Initially, the block and the board are at rest. Coefficient of kinetic friction between the small block and the board is μ_k. At time $t = 0$, the block is given an initial velocity $\vec{\mathbf{v}}_0$ as shown.

(a) (5 Pts.) Draw a free-body diagram for each object.

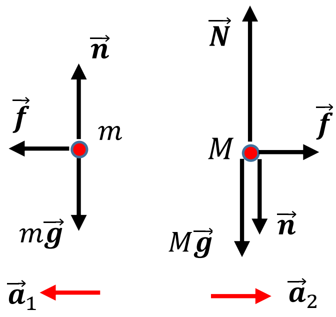
(b) (5 Pts.) What are the accelerations of the block and of the board relative to the horizontal surface for $t > 0$?



(c) (5 Pts.) What should be the magnitude of $\vec{\mathbf{v}}_0$
if the block is to stop moving relative to the board after it travels a distance L relative to the board?

(d) (5 Pts.) How much energy will be lost by the system when the small block stops sliding on the board?

Solution:



(b) Applying Newton's second law to the block, we have

$-f = ma_1, n - mg = 0$. Since $f = \mu_k n = \mu_k mg$, we get

$$a_1 = -\mu_k g\,.$$

Applying Newton's second law to the board in the horizontal direction, we have

$$f = Ma_2 \quad \rightarrow \quad a_2 = \frac{m}{M}\mu_k g\,.$$

(c) Velocities of the block and of the board respectively relative to the horizontal surface is

$$v_1 = v_0 - \mu_k g t\,, \qquad v_2 = \frac{m}{M}\mu_k g\, t\,.$$

The block will stop moving relative to the board when their velocities relative to the horizontal surface are equal. The time at which this occurs is found as

$$v_0 - \mu_k g t = \frac{m}{M}\mu_k g\, t \quad \rightarrow \quad t = \frac{v_0}{\left(1 + \frac{m}{M}\right)\mu_k g} = \frac{M v_0}{(M+m)\mu_k g}\,, \qquad v_1(t) = v_2(t) = \frac{m v_0}{M+m}.$$

If the block is to stop relative to the board at the other end of the board, their relative displacement must be equal to L at the time found above. So,

$$x_1 - x_2 = v_0 t - \frac{1}{2}\mu_k g t^2 - \frac{1}{2}\frac{m}{M}\mu_k g t^2 = v_0 t - \frac{1}{2}\left(1 + \frac{m}{M}\right)\mu_k g t^2 = \frac{v_0^2}{2\left(1 + \frac{m}{M}\right)\mu_k g} = \frac{M v_0^2}{2(M+m)\mu_k g}.$$

Note that the same result can be obtained using the kinematic relation $v_0^2 = 2a_{12}(x_1 - x_2)$.

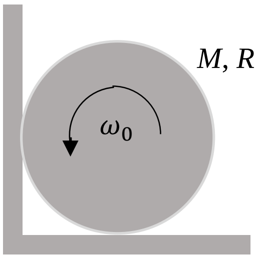
$$x_1 - x_2 = L \quad \rightarrow \quad v_0 = \sqrt{2\left(1 + \frac{m}{M}\right)\mu_k g L}\,.$$

(d)

$$E_1 = \frac{1}{2}m v_0^2\,, \qquad E_2 = \frac{1}{2}(M+m)\frac{m^2 v_0^2}{(M+m)^2} = \frac{m^2 v_0^2}{2(M+m)} \quad \rightarrow \quad \Delta E = \frac{1}{2}\left(\frac{Mm}{M+m}\right)v_0^2\,.$$

PHYS 101 Final Exam Question 3

A cylindrical disk of mass M and radius R ($I_{CM} = \frac{1}{2}MR^2$) skids against both the horizontal and vertical surfaces of a corner as shown in the figure. The coefficient of kinetic friction between the disk and both surfaces is μ_k. The initial angular speed of the disk at the instant it is placed in the corner is ω_0, and its axis of rotation does not move while it rotates.

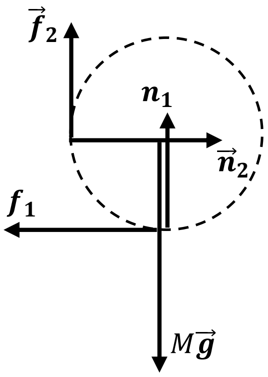


(a) (5 Pts.) Draw a free body diagram for the disk.

(b) (15 Pts.) Find the total number of revolutions the disk makes before stopping.

Solution:

(a)



(b) We have $\vec{\boldsymbol{a}} = 0$ and $\alpha \neq 0$. Therefore, $\sum \vec{\boldsymbol{F}} = 0$, and $\sum \vec{\boldsymbol{\tau}} \neq 0$. Newton's second law for the translational motion in the horizontal and the vertical directions are written respectively as

$n_2 - f_1 = 0$, and $n_1 + f_2 - Mg = 0$. We know that $f_1 = \mu_k n_1$ and $f_2 = \mu_k n_2$. Hence, we have

$-\mu_k n_1 + n_2 = 0$ and $n_1 + \mu_k n_2 = Mg$. Solving these two equations for n_1 and n_2, we find

$$n_1 = \frac{Mg}{1+\mu_k^2}, \qquad n_2 = \frac{\mu_k Mg}{1+\mu_k^2} \quad \rightarrow \quad f_1 = \frac{\mu_k Mg}{1+\mu_k^2}, \qquad f_2 = \frac{\mu_k^2 Mg}{1+\mu_k^2}.$$

Newton's second law for the rotational motion is written as $\sum \tau = -RF_{f1} - RF_{f2} = I\alpha$. Hence, the angular acceleration of the disk is found as

$$\alpha = -\frac{R}{I}(f_1 + f_2) = \frac{2}{MR}\left(\frac{\mu_k Mg}{1+\mu_k^2} + \frac{\mu_k^2 Mg}{1+\mu_k^2}\right) \quad \rightarrow \quad \alpha = -\frac{2g}{R}\left(\frac{\mu_k + \mu_k^2}{1+\mu_k^2}\right).$$

Since $\omega^2 - \omega_0^2 = 2\alpha\theta$, and $\omega = 0$ when the disk stops, we find

$$\theta = -\frac{\omega_0^2}{2\alpha} = \frac{R\omega_0^2}{4g}\left(\frac{1+\mu_k^2}{\mu_k + \mu_k^2}\right).$$

Hence, total number of revolutions the disk makes before stopping is

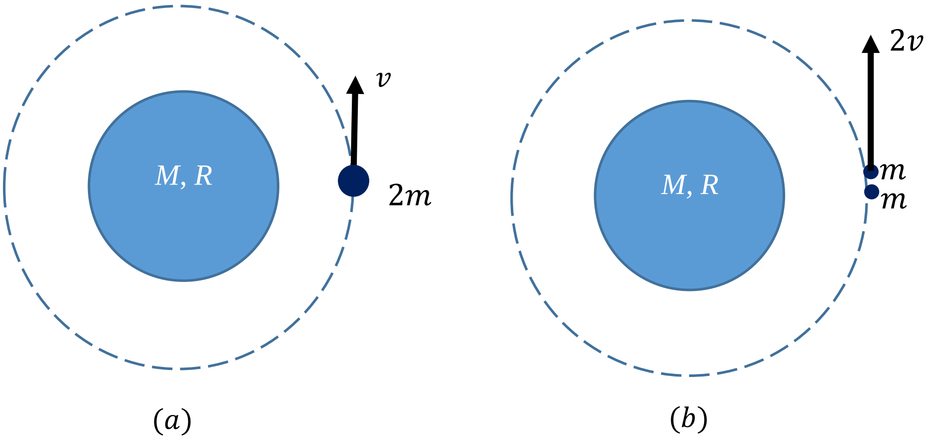
$$\text{Number of Revolutions} = \frac{\theta}{2\pi} = \frac{R\omega_0^2}{8\pi g}\left(\frac{1+\mu_k^2}{\mu_k + \mu_k^2}\right).$$

PHYS 101 Final Exam Question 4

A satellite of mass $2m$ is in a circular orbit with speed v, around a planet of mass M and radius R. The planet has no atmosphere (Figure (a)). Answer the following in terms of M, m, R, G and v.

(a) (10 Pts.) Suddenly an explosion inside the satellite breaks it up into two pieces of equal mass (Figure (b)). One of the pieces leave the explosion with speed $2v$ in the direction of velocity before the explosion. Find the speed of the other piece when it collides with the planet.

(b) (10 Pts.) What is the total mechanical energy and the angular momentum of the piece with speed $2v$ after the explosion? How far away from the center of the planet does this piece go, and what will be its final speed?



Solution:

(a) Radius of the circular orbit and the speed of the satellite in that orbit are related by the expression $r = GM/v^2$.

Momentum is conserved during the explosion, meaning that

$$p_1 = p_2 \quad \rightarrow \quad (2m)v = m(2v) + mv' \quad \rightarrow \quad v' = 0,$$

where v' is the speed of the second piece after the explosion. Hence, the second piece will start to fall down towards the planet with zero initial speed. Total mechanical energy is conserved during the fall. This means

$$E_1 = E_2 \quad \rightarrow \quad -G\frac{Mm}{r} = \frac{1}{2}mv_2^2 - G\frac{Mm}{R} \quad \rightarrow \quad v_2 = \sqrt{2\left(\frac{GM}{R} - v^2\right)}.$$

(b) The total mechanical energy and the angular momentum of the first piece immediately after the explosion are

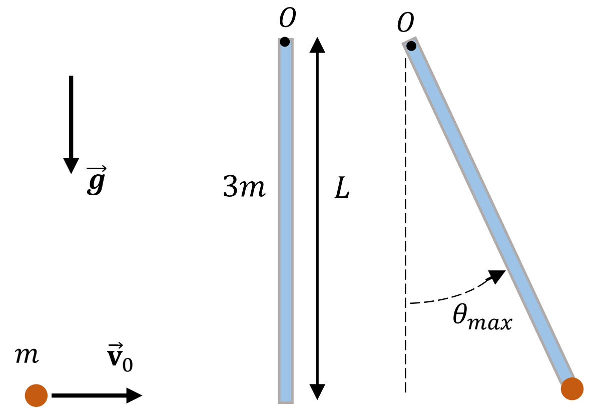
$$E_1 = \frac{1}{2}m(2v)^2 - G\frac{Mm}{r} = mv^2\,, \qquad L_1 = m(2v)r = 2G\frac{Mm}{v}\,,$$

and both will be conserved. Because $E_1 > 0$, this piece will escape, and reach an infinite distance away from the planet. Since it will have no potential energy when it escapes, all its energy will be in kinetic form. Hence, its final speed v_{1f} is found as

$$E_1 = \frac{1}{2}mv_{1f}^2 = mv^2 \quad \rightarrow \quad v_{1f} = \sqrt{2}v\,.$$

PHYS 101 Final Exam Question 5

A thin uniform rod of mass $3m$ and length L is pivoted at its top end with a frictionless axle, and is at rest hanging vertically. A small object with mass m and initial velocity $\vec{\mathbf{v}}_0$ collides with the rod, and sticks to the free end, as shown in the figure. After the collision, the rod swings up until it makes a maximum angle θ_{max} with the vertical. Answer the following questions in tems of the parameters $m, L, g,$ and v_0. (The moment of inertia of a rod with mass M and length L about its center of mass is $I_{\rm CM} = ML^2/12$.)



(a) (10 Pts.) What is the angular speed of the rod immediately after the collison?

(b) (10 Pts.) Following the collision, the rod-mass system will be oscillating about the axle. What is the period of small oscillation?

Solution:

(a) Angular momentum of the system with respect to the pivot is conserved during the collision. So

$$L_2 = L_1 \quad \rightarrow \quad I\omega = mv_0L.$$

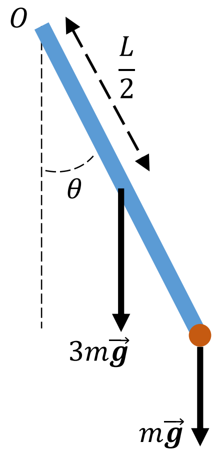
Moment of inertia of the rod with the object stuck to its end is

$$I = I_{\rm rod} + I_{\rm obj} = I_{\rm CM} + 3m\left(\frac{L}{2}\right)^2 + mL^2 \quad \rightarrow \quad I = 2mL^2\,.$$

Hence,

$$\omega = \frac{mv_0L}{I} \quad \rightarrow \quad \omega = \frac{v_0}{2L}.$$

(b)



$$\tau = I\alpha \quad \rightarrow \quad 2mL^2\frac{d^2\theta}{dt^2} = -3mg\left(\frac{L}{2}\right)\sin\theta - mgL\sin\theta = -\frac{5}{2}mgL\sin\theta\,.$$

For small oscillations, using $\sin\theta \approx \theta$, the equation of motion is found to be

$$\frac{d^2\theta}{dt^2} + \left(\frac{5g}{4L}\right)\theta = 0\,.$$

Angular frequency of small oscillatios can be identified, and we get

$$\omega = \sqrt{\frac{5g}{4L}} \quad \rightarrow \quad T = 2\pi\sqrt{\frac{4L}{5g}}.$$